

# Fourier Series Expansion

Deepesh K P

There are many types of series expansions for functions. The Maclaurin series, Taylor series, Laurent series are some such expansions. But these expansions become valid under certain strong assumptions on the functions (those assumptions ensure convergence of the series). Fourier series also express a function as a series and the conditions required are fairly good and suitable when we deal with signals.

Suppose  $f$  is a real valued function from  $\mathbb{R}$  to  $\mathbb{R}$ . In this note, we deal with the following three questions:

- When does  $f$  has a Fourier series expansion?
- How we find the expansion?
- What are the main properties of this expansion?

## 1 Existance of a Fourier series expansion:

There are three conditions which guarantees the existance of a valid Fourier series expansion for a given function. These conditions are collectively called the *Dirichlet conditions*:

1.  $f$  is a periodic function on  $\mathbb{R}$ . This means that there exists a period  $T \geq 0$  such that

$$f(x) = f(x + T) \text{ for all } x \in \mathbb{R}.$$

2.  $f$  has only a finite number of maxima and minima in a period.
3.  $f$  has atleast a finite number of discontinuous points inside a period.
4.  $f$  is integrable over the period of the function.

It should be noted that the second and third conditions are satisfied by many real valued functions that we deal with, inside any finite interval. But periodicity is a condition that is satisfied by very few functions, for example, *constant function*, *sine*, *cos*, *tan* and their combinations. But we can consider any function defined on a finite interval  $[a, b]$  (or  $(a, b)$ ) as a periodic function on  $\mathbb{R}$  by thinking that the function is extended to  $\mathbb{R}$  by repeating the values in  $[a, b]$  to the remaining part of  $\mathbb{R}$ . Thus

| Most of the functions, that we commonly use, defined |  
| on finite intervals can be expanded as Fourier series |

Figure

## 2 Derivation of Fourier series expansion of a function defined in $[-\pi, \pi]$ :

In Fourier series expansion, we would like to write the function as a series in sine and cosine terms in the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

For finding the above unknown co-efficients  $a_0, a_n$  and  $b_n$  in the Fourier series expansion of a function, one need to recall the value of certain integrals:

1.  $\int_{-\pi}^{\pi} \sin mx \, dx = 0$  for any integer  $m$ .
2.  $\int_{-\pi}^{\pi} \cos mx \, dx = 0$  for any integer  $m$ .
3.  $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$  for **any** integers  $m$  and  $n$ .
4.  $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$  for integers  $m \neq n$ .
5.  $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$  for integers  $m \neq n$ .
6.  $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi$  when the integers  $m = n$ .
7.  $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi$  when the integers  $m = n$ .

[All the above integrals easily follow by evaluating using integration by parts]

Now suppose  $f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos jx + b_j \sin jx$ .

**To find  $a_0$ :**

Observe that

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{j=1}^{\infty} \left( a_j \int_{-\pi}^{\pi} \cos jx \, dx + b_j \int_{-\pi}^{\pi} \sin jx \, dx \right) \\ &= \frac{a_0}{2} 2\pi + \sum_{j=1}^{\infty} (0 + 0) \end{aligned}$$

This implies that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

**To find  $a_n$ :**

Observe that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cos nx \, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{j=1}^{\infty} \left( a_j \int_{-\pi}^{\pi} \cos nx \cos jx \, dx + b_j \int_{-\pi}^{\pi} \cos nx \sin jx \, dx \right) \\ &= \frac{a_0}{2} 0 + a_n \pi + \sum_{j=1}^{\infty} b_j 0\end{aligned}$$

This implies that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

**To find  $b_n$ :**

Observe that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \sin nx \, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx \, dx + \sum_{j=1}^{\infty} \left( a_j \int_{-\pi}^{\pi} \sin nx \cos jx \, dx + b_j \int_{-\pi}^{\pi} \sin nx \sin jx \, dx \right) \\ &= \frac{a_0}{2} 0 + \sum_{j=1}^{\infty} a_j 0 + b_n \pi.\end{aligned}$$

This implies that

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Thus

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

[This expansion is valid at all those points  $x$ , where  $f(x)$  is continuous.]

**Note:** Note that the above mentioned results hold when we take any  $2\pi$  length intervals [This is because  $\int_c^{c+2\pi} \sin mx \, dx = 0, \dots$  are true for any  $c$ ].

**Result:** So whenever we take a function  $f$  defined from  $[c, c + 2\pi]$  (any interval of length  $2\pi$ ) to  $\mathbb{R}$ , satisfying the Dirichlet conditions, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx$$

### 3 Derivation of Fourier series expansion of a function defined in an arbitrary period $[a, b]$ :

Now suppose that  $f(x)$  is defined in an arbitrary interval  $[a, b]$  and satisfy the Dirichlet conditions. Let us take  $\frac{b-a}{2} = l$ , half the length of the interval. Now define the new variable

$$z = \frac{\pi}{l}x.$$

By this simple transformation, we can convert functions on any finite interval (say,  $[a, b]$ ) to functions in the new variable  $z$ , whose domain is an interval of  $2\pi$  length. This is because

$$\begin{aligned} x = a &\Rightarrow z = \frac{\pi}{l}a \text{ and} \\ x = b &\Rightarrow z = \frac{\pi}{l}b = \frac{2\pi}{b-a}(b-a+a) = 2\pi + \frac{\pi}{l}a. \end{aligned}$$

Thus when the variable  $x$  in  $f(x)$  moves from  $a$  to  $b$ , the new variable  $z$  in the new function  $F(z)$  (which is the same function  $f$  in the new variable) moves from  $c$  to  $c+2\pi$ , where  $c = \frac{\pi}{l}a$ . Hence the Fourier series expansion is applicable for  $F(z)$ . Thus

$$f(x) = F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + b_n \sin nz,$$

where

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} F(z) dz$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} F(z) \cos nz dz$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} F(z) \sin nz dz$$

and changing back to the original variable  $x$  (note that  $dz = \frac{\pi}{l}dx$ ), we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x,$$

where

$$a_0 = \frac{1}{l} \int_a^b f(x) dx$$

$$a_n = \frac{1}{l} \int_a^b f(x) \cos \frac{n\pi}{l} x dx$$

$$b_n = \frac{1}{l} \int_a^b f(x) \sin \frac{n\pi}{l} x dx,$$

which is the general form of Fourier series expansion for functions on any finite interval. Also note that this is applicable to the first case of our discussion, where we need to take  $a = -\pi$ ,  $b = \pi$ ,  $l = \pi$  and then everything becomes the same as in the previous section.

### 3.1 Illustration

We now take a simple problem to demonstrate the evaluation of Fourier series.

Consider the function  $f$  defined by

$$f(x) = \begin{cases} -10 & \text{if } -2 \leq x \leq -1, \\ x & \text{if } -1 < x < 1, \\ 10, & \text{if } 1 \leq x \leq 2. \end{cases}$$

We shall find the Fourier series expansion of this function. Here, note that the length of the interval is 4. So  $2l = 4$  and  $l = 2$ . We need to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} x + b_n \sin \frac{n\pi}{2} x,$$

where

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi}{2} x dx$$

$$b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi}{2} x dx,$$

Now

$$\begin{aligned} a_0 &= \frac{1}{2} \left( \int_{-2}^{-1} -10 dx + \int_{-1}^1 x dx + \int_1^2 10 dx \right) \\ &= \frac{1}{2} (-10 + 0 + 10) = 0 \end{aligned}$$

$$a_n = \frac{1}{2} \left( \int_{-2}^{-1} -10 \cos \frac{n\pi}{2} x dx + \int_{-1}^1 x \cos \frac{n\pi}{2} x dx + \int_1^2 10 \cos \frac{n\pi}{2} x dx \right)$$

$$\begin{aligned}
&= \frac{1}{2}(-10 \int_{-2}^{-1} \cos \frac{n\pi}{2} x dx + \int_{-1}^1 x \cos \frac{n\pi}{2} x dx + 10 \int_1^2 \cos \frac{n\pi}{2} x dx) \\
&= \frac{1}{2}(\frac{20}{n\pi}(\sin \frac{n\pi}{2} - \sin n\pi) + 0 + \frac{20}{n\pi}(\sin n\pi - \sin \frac{n\pi}{2})) = 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2}(\int_{-2}^{-1} -10 \sin \frac{n\pi}{2} x dx + \int_{-1}^1 x \sin \frac{n\pi}{2} x dx + \int_1^2 10 \sin \frac{n\pi}{2} x dx) \\
&= \frac{1}{2}\{\frac{20}{n\pi}(\cos(\frac{n\pi}{2}) - \cos(n\pi)) + 2[\frac{-2}{n\pi} \cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2} \sin(\frac{n\pi}{2}) - 0] - \frac{20}{n\pi}(\cos(n\pi) - \cos(\frac{n\pi}{2}))\} \\
&= \frac{18}{n\pi} \cos(\frac{n\pi}{2}) - \frac{20}{n\pi} \cos(n\pi) + \frac{4}{n^2\pi^2} \sin(\frac{n\pi}{2}).
\end{aligned}$$

So when  $n = 1 \Rightarrow b_1 = \frac{4}{\pi^2}$ ,  $n = 2 \Rightarrow b_2 = \frac{19}{\pi}, \dots$

Thus the Fourier expansion of  $f(x)$  is

$$\begin{aligned}
f(x) &= \frac{0}{2} + 0 \cos \frac{\pi}{2} x + \frac{4}{\pi^2} \sin \frac{\pi}{2} x + 0 \cos \frac{2\pi}{2} x + \frac{19}{\pi} \sin \frac{2\pi}{2} x + \dots \\
&= \frac{4}{\pi^2} \sin \frac{\pi}{2} x + \frac{19}{\pi} \sin \frac{2\pi}{2} x + \dots,
\end{aligned}$$

which is valid at all points in  $[-2, 2]$  except at  $-1$  and  $1$ , since the function is continuous at all points except  $-1$  and  $1$ . When  $x = -1$ , the sum of the series will be equal to the value  $\frac{-10+1}{2} = -5.5$  and at  $x = 1$ , it is  $\frac{10+1}{2} = 5.5$

### 3.2 Some special cases:

Suppose the function is an odd/even function in a symmetric interval  $[-c, c]$ . That is  $f(-x) = f(x)$  for all  $x \in \mathbb{R}$  [ $\Rightarrow$  Even] or  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$  [ $\Rightarrow$  Odd]. Then from the results,

$$\int_{-c}^c f(x) dx = 0 \text{ when } f(x) \text{ is odd}$$

$$\int_{-c}^c f(x) dx = 2 \int_0^c f(x) dx \text{ when } f(x) \text{ is even ,}$$

we have some specialities in the expansions.

#### 3.2.1 Even functions:

Suppose  $f(x)$  is even in a domain  $[-c, c]$ . Then it can be observed that  $b_n = 0$  and so the Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{c} x$$

where

$$\begin{aligned}
a_0 &= \frac{2}{c} \int_0^c f(x) dx \\
a_n &= \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi}{c} x dx
\end{aligned}$$

### 3.2.2 Odd functions:

Similarly when  $f(x)$  is odd in a domain  $[-c, c]$ . Then  $a_0 = a_n = 0$  and the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{c} x$$

where

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi}{c} x dx$$

**Note:** If you observe carefully, in the illustration problem, the function is actually odd and the domain is  $[-2, 2]$ . Note that the Fourier series contains only sine terms. But the converse is not true in general. That is, even if the Fourier series contains only sine terms, the function may not be odd!

[For proving the above two cases, one should recall that the product of an odd and an even function is always odd and when both functions are even or both odd then the product is always even.]

### 3.3 Evaluation of series

Fourier series can be used for evaluating the sum of certain numerical series related with it. For each value of  $f(x_0)$ , where  $x_0$  is a continuous point of the function, we get a series by putting the value  $x_0$  on both sides of the function. Care should be given if we substitute a discontinuous point as then instead of  $f(x_0)$ , we should use the average value of the left and right limit at  $x_0$

#### 3.3.1 Illustration

Suppose  $f(x) = x^2$ ,  $-\pi < x < \pi$ . We shall find the Fourier series of this function and use it to evaluate the sum of the series  $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

Observe that the function is an even function in the symmetric interval  $(-\pi, \pi)$ . So we immediately get  $b_n = 0$  for all  $n$ . So we need to find only  $a_0, \&a_n$ .

$$a_0 = \int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^2$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = (-1)^n \frac{4}{n^2}$$

Thus the Fourier series becomes

$$x^2 = \frac{\pi^2}{3} + -1 \cdot \frac{4}{1^2} \cos x + 1 \cdot \frac{4}{2^2} \cos 2x + -1 \cdot \frac{4}{3^2} \cos 3x + \dots$$

Now to find the sum of the given series, take  $x = 0$ , which is a continuous point. Substituting on both sides

$$0 = \frac{\pi^2}{3} + -4 \left( \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right),$$

which gives

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}.$$

## 4 Fourier Sine Series and Cosine Series:

Sometimes we may need to expand a given function as a series in only cosine terms or only sine terms. Such series are called *cosine series* and *sine series* respectively. This can be done easily when the given function is defined on an interval like  $[0, c]$  (that is a positive side interval) or  $[-c, 0]$  (negative side interval). Hence such series are also referred as *Half range series expansion*.

Suppose  $f(x)$  is defined in the interval  $[0, c]$ . Then one can **think** of the extended function as either even or odd (depending on the need; i.e., cosine series  $\Rightarrow$  even and sine series  $\Rightarrow$  odd) into the interval  $[-c, c]$ .

### Extension as even function - Cosine series:

Suppose we think of  $f(x)$  as even in the interval  $[-c, c]$ . Now expand this function. Obviously the series you obtain will have only constant and cosine terms. This is the cosine series of the function  $f(x)$ . That is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{c} x$$

where

$$a_0 = \frac{2}{c} \int_0^c f(x) dx$$
$$a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi}{c} x dx$$

This type of a series is valid only in the domain  $[0, c]$  as the function on the other side was just an extension (not a part of the original function).

### Extension as odd function - Sine series:

Similarly when  $f(x)$  is thought as an extended odd function in a domain  $[-c, c]$ . Then the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{c} x$$

where

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi}{c} x dx$$

Note: For doing problems one need not do the extension to  $[-c, c]$ . Just applying the formula is fine.

Note: Thus a function defined in  $[0, c]$  will have three types of expansions:

- a Fourier series expansion
- a Cosine series expansion
- a Sine series expansion.



## 4.1 Illustration

Consider the function  $f(x) = x$ ,  $0 < x < \pi$ . We shall find the Fourier cosine series of this function.

First, considering as an even function in  $(-\pi, \pi)$ , we get the Fourier cosine series as  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Note that  $l = \pi$  here. Let us find  $a_0$  and  $a_n$ .

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx \\ &= \frac{2}{\pi} \frac{\pi^2}{2} = \pi^2. \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \\ &= \frac{2}{\pi} \left[ \frac{(-1)^2}{n^2} - \frac{1}{n^2} \right] \end{aligned}$$

Thus the Fourier cosine series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[ \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

In a similar way, one can apply the formula to find the Fourier sine series of the function. Note that the same function also has got a Fourier series expansion, where you need to apply the formula by taking  $l = \frac{\pi}{2}$ .

## 5 Practical Harmonic Analysis

In many practical problems, we get the data at discrete points of time. That is, the function will not be given explicitly but the function values will be given at, say, regular intervals of the domain. For example,

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	$\pi$
$f(x)$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	-1

Here the function is actually  $\sin x$ , but given at discrete points,  $0, \frac{\pi}{4}, \frac{\pi}{2}, 3\frac{\pi}{4}, \pi$ . With these point values, how to calculate the Fourier series?

### 5.1 The Trapezoidal Rule of Integration

This is a rule used for finding the definite integral of a function, which is given at equally spaced, discrete points. Suppose the values of a function are given as in

$x$	$x_0$	$x_1$	$x_2$	$\dots$	$x_{n-1}$	$x_n$
$f(x)$	$y_0$	$y_1$	$y_2$	$\dots$	$y_{n-1}$	$y_n$

were the points  $x_0, x_1, \dots, x_n$  are equally spaced; i.e.,  $x_{i+1} - x_i = h$ , a constant, for all  $i$ . Then the Trapezoidal rule says that

$$\int_{x_0}^{x_n} f(x) \, dx = \frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

### 5.1.1 Example

Calculate  $\int_1^4 f(x)dx$  where  $f(x)$  is given at

x:	1	1.5	2	2.5	3	3.5	4
f(x):	2	4.875	10	18.125	30	46.375	68

Solution: Note that  $h = 0.5$  here. Now from trapezoidal rule,

$$\int_1^4 f(x)dx = \frac{0.5}{2} (2 + 2[4.875 + 10 + 18.125 + 30 + 46.375] + 68) = 72.1875.$$

## 5.2 Calculating the Fourier series from a discrete data

Suppose that the values of a function are given as in

$x$	$x_0$	$x_1$	$x_2$	$\dots$	$x_{n-1}$	$x_n$
$f(x)$	$y_0$	$y_1$	$y_2$	$\dots$	$y_{n-1}$	$y_n$

Here the length of the interval is  $x_n - x_0$  and so  $l = \frac{x_n - x_0}{2}$ . The spacing between the points is  $h = x_{i+1} - x_i$ . The Fourier series of the function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x,$$

where

$$a_0 = \frac{1}{l} \int_{x_0}^{x_n} f(x) dx$$

$$a_n = \frac{1}{l} \int_{x_0}^{x_n} f(x) \cos \frac{n\pi}{l} x dx$$

$$b_n = \frac{1}{l} \int_{x_0}^{x_n} f(x) \sin \frac{n\pi}{l} x dx,$$

Since  $f(x)$  is not explicitly given here, we need to use the Trapezoidal rule to find  $a_0, a_n$  &  $b_n$ :

$$\begin{aligned} a_0 &= \frac{1}{l} \int_{x_0}^{x_n} f(x) dx \\ &= \frac{1}{l} \frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n] \\ a_n &= \frac{1}{l} \int_{x_0}^{x_n} f(x) \cos \frac{n\pi}{l} x dx \\ &= \frac{1}{l} \frac{h}{2} \left[ y_0 \cos \frac{n\pi}{l} x_0 + 2 \left( y_1 \cos \frac{n\pi}{l} x_1 + y_2 \cos \frac{n\pi}{l} x_2 + \dots + y_{n-1} \cos \frac{n\pi}{l} x_{n-1} \right) + y_n \cos \frac{n\pi}{l} x_n \right] \\ b_n &= \frac{1}{l} \int_{x_0}^{x_n} f(x) \sin \frac{n\pi}{l} x dx \\ &= \frac{1}{l} \frac{h}{2} \left[ y_0 \sin \frac{n\pi}{l} x_0 + 2 \left( y_1 \sin \frac{n\pi}{l} x_1 + y_2 \sin \frac{n\pi}{l} x_2 + \dots + y_{n-1} \sin \frac{n\pi}{l} x_{n-1} \right) + y_n \sin \frac{n\pi}{l} x_n \right] \end{aligned}$$

In problems, you can not find all the  $a_n$ s. So find  $a_0, a_1, a_2, b_1, b_2$  and substitute in the Fourier series formula.

### 5.2.1 Illustration

Find the Fourier series of the function from the data given below:

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	$\pi$
$f(x)$	1	2	3	4	5

Here the interval under consideration is  $(0, \pi)$ , which is of length  $\pi$ . Hence  $l = \frac{\pi}{2}$  and  $h = \frac{\pi}{4}$ . The Fourier series of the function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos 2nx + b_n \sin 2nx,$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos 2nx dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin 2nx dx,$$

Now we calculate  $a_0, a_1, a_2, b_1, b_2$  using the Trapezoidal rule.

$$\begin{aligned} a_0 &= \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \left\{ \frac{\pi}{8} [1 + 2(2 + 3 + 4) + 5] \right\} = 6 \\ a_1 &= \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \cos 2x dx \\ &= \frac{2}{\pi} \left\{ \frac{\pi}{8} \left[ 1 \cos 2 \cdot 0 + 2 \left( 2 \cos 2 \cdot \frac{\pi}{4} + 3 \cos 2 \cdot \frac{\pi}{2} + 4 \cos 2 \cdot 3 \frac{\pi}{4} \right) + 5 \cos 2\pi \right] \right\} \\ &= \frac{1}{4} [1 + 2(2 \cdot 0 + 3 \cdot -1 + 4 \cdot 0) + 5 \cdot 1] = 0 \\ a_2 &= \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \cos 4x dx \\ &= \frac{2}{\pi} \left\{ \frac{\pi}{8} \left[ 1 \cos 4 \cdot 0 + 2 \left( 2 \cos 4 \cdot \frac{\pi}{4} + 3 \cos 4 \cdot \frac{\pi}{2} + 4 \cos 4 \cdot 3 \frac{\pi}{4} \right) + 5 \cos 4\pi \right] \right\} \\ &= \frac{1}{4} [1 + 2(2 \cdot -1 + 3 \cdot 1 + 4 \cdot -1) + 5 \cdot 1] = 0 \\ b_1 &= \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \sin 2x dx \\ &= \frac{2}{\pi} \left\{ \frac{\pi}{8} \left[ 1 \sin 2 \cdot 0 + 2 \left( 2 \sin 2 \cdot \frac{\pi}{4} + 3 \sin 2 \cdot \frac{\pi}{2} + 4 \sin 2 \cdot 3 \frac{\pi}{4} \right) + 5 \sin 2\pi \right] \right\} \\ &= \frac{1}{4} [0 + 2(2 \cdot 1 + 3 \cdot 0 + 4 \cdot -1) + 5 \cdot 0] = -1 \\ b_2 &= \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \sin 4x dx \\ &= \frac{2}{\pi} \left\{ \frac{\pi}{8} \left[ 1 \sin 4 \cdot 0 + 2 \left( 2 \sin 4 \cdot \frac{\pi}{4} + 3 \sin 4 \cdot \frac{\pi}{2} + 4 \sin 4 \cdot 3 \frac{\pi}{4} \right) + 5 \sin 4\pi \right] \right\} \\ &= \frac{1}{4} [1 \cdot 0 + 2(2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0) + 5 \cdot 0] = 0 \end{aligned}$$

Thus the Fourier series expansion of the function is

$$f(x) = \frac{0}{2} + 0.\cos 2x + 0.\cos 4x + \dots + -1.\sin 2x + 0.\sin 4x + \dots$$

## 6 Problems

1. Obtain the Fourier series representation of  $f(x) = \frac{1}{4}(\pi - x)^2$ ,  $0 < x < 2\pi$ .

[Hint:  $l = \pi$ ,  $a_0 = \frac{\pi^2}{6}$ ,  $a_n = \frac{1}{n^2}$ ,  $b_n = 0$  ]

2. Expand  $f(x) = x \sin x$ ,  $0 \leq x \leq 2\pi$  as a Fourier series.

[Hint: Even,  $l = \pi$ ,  $b_n = 0$ ,  $a_0 = -2$ ,  $a_n = \frac{2}{n^2-1}$ , if  $n \neq 1$ ,  $a_1 = \frac{-1}{2}$ ]