Vector spaces

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One can observe that if we consider $\mathcal{M}_{2\times 3}$, the collection of all 2×3 matrices, there are two operations defined: the matrix addition - defined between two matrices - and the scalar multiplication - between a matrix and a number. These operations enjoy lots of properties. Motivated by those properties, we define the concept of a vector space:

Definition 0.1. *A non empty set V together with two operations - vector addition $'+'$ and scalar multiplication $'.'$ - is called a vector space if it satisfy the following properties (here we assume that u, v, w are arbitrary elements of V and α, β are arbitrary real numbers):*

1. $u + v \in V$
2. $u + v = v + u$
3. $u + (v + w) = (u + v) + w$
4. *there is an element* 0 *in* V *such that* $0 + u = u$
5. *there is an element* $-u$ *in* V *such that* $u + (-u) = 0$
6. $\alpha.u \in V$
7. $(\alpha\, +\, \beta).u = \alpha.u + \beta.u$
8. $\alpha.(u + v) = \alpha.u + \alpha.v$
9. $(\alpha\, \beta).u = \alpha\, (\beta.u) = \beta\, (\alpha.u)$
10. $1.u = u$

The elements of a vector space V are called *vectors* and the real numbers $\alpha,\ \beta, \ldots$ are called *scalars*. A set should satisfy all the 10 properties of the operations if it is to be called a vector space. If any one of these 10 properties is not satisfied, the set can not be a vector space.

Example 0.2. (1) The set of all $m \times n$ matrices denoted by $\mathcal{M}_{m\times n}$, for each $m, n \in \mathbb{N}$, with usual matrix addition and scalar multiplication.

(2) The set $\mathbb{R}^n$ of all $n-$ tuples (one may think of this as a special case of [2.] - the row matrix) with usual addition of n-tuples and multiplication of an $n-$tuple by a number.

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\\ \alpha\, (x_1, x_2, \ldots, x_n) &= (\alpha\, x_1, \alpha\, x_2, \ldots, \alpha\, x_n)\end{aligned}$$

(3) The set of all real numbers $\mathbb{R}$ (special case of [2], when $n = 1$)

(4) The set $\mathcal{P}_n$ of all polynomials of order less than or equal n with usual polynomial addition and multiplication of polynomial with numbers.

$$\begin{aligned}(a_0 + a_1 t + \ \ldots + a_n t^n) + (b_0 + b_1 t + \ \ldots + b_n t^n) &= (a_0 + b_0) + (a_1 + b_1)t + \ \ldots + (a_n + b_n)t^n\\ \alpha\, (a_0 + a_1 t + \ \ldots + a_n t^n) &= \alpha\, a_0 + \alpha\, a_1 t + \ \ldots + \ \alpha\, a_n t^n\end{aligned}$$

(5) The set of all polynomials $\mathcal{P}$, as an extended case of [4].

(6) The set of all real valued functions defined on $[a, b]$ with

$$\begin{aligned}(f + g)(x) &= f(x) + g(x), \quad x \in [a, b]\\ (\alpha\, f)(x) &= \alpha.\, f(x), \quad x \in [a, b].\end{aligned}$$

All these vector spaces are standard and are commonly used in the study of vector spaces. Now we give some examples of sets which are not vector spaces.

Example 0.3. Let V be the set of all matrices of the form

$$\left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

with usual addition and scalar multiplication. Here one can observe that V does not satisfy the properties (1) and (6). To see that (1) is not satisfied,

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 2 \end{bmatrix} \notin V$$

So, eventhough V satisfy the properties (2), (3), (7), (8), (9), and (10), it is not a vector space.

Following is the concept of a subspace of a vector space.

Definition 0.4. *Let V be a vector space. A subset W of V is said to be a subspace if W itself is a vector space under the induced operations.*

This means that a vector space sitting inside a bigger vector space is called a subspace. Of course, the operations in the subspace should be the same as that of the bigger space.

Example 0.5. Let W be the set of all matrices of the form

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

with addition and scalar multiplication as in $\mathcal{M}_{2\times2}$. Obviously this collection is a subset of $\mathcal{M}_{2\times2}$. One can easily check that the properties (1) to (10) of a vector space are satisfied by this W. Hence W is a subspace of V.

It is not always needed to check all the 10 properties for proving that a subset is a subspace. One can observe that the properties [2], [3], [7], [8], [9] and [10] are heriditory properties (it means that if the operation is the same, such a property of a bigger space carry over to a smaller space). Also, the following result helps to say that, in a vector space, properties [4] and [5] follows from other properties.

Theorem 0.6. *Let V be a vector space. Then for each $u \in V$,*

$$0.\, u = 0,$$

the identity of V, and

$$-1.\, u = -u,$$

the inverse of u.

Proof. We have

$$0.\, u = (0 + 0).\, u = 0.\, u + 0.\, u.$$

This shows that $0.\, u$ is the identity. That is $0.\, u = 0$. Now

$$u + -1.\, u = 1.\, u + -1.\, u = (1 + -1).\, u = 0.\, u = 0,$$

which shows that $-1.\, u$ is the inverse of u. □

Thus, in order to check a given subset W of V is a subspace or not, it is enough to check properties [1] and [6] only, provided the operations in W are not altered.

Theorem 0.7. *A subset W of a vector space V is a subspace if and only if*

i. *for every $u, v \in W$, $u + v \in W$.*
ii. *for every $\alpha \in \mathbb{R}$ and $u \in W$, $\alpha.\, u \in W$.*

Given a subset of a vector space, one can use Theorem 0.7 to check if it is a subspace or not. One should note that this theorem can not be used if the operations given in the subset are mentioned explicitely to be different from that of the vector space.

Example 0.8. Let W be the set of all matrices of the form

$$\left\{ \begin{bmatrix} a & 0 \\ b & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

with addition and scalar multiplication as in $\mathcal{M}_{2\times2}$. Obviously this collection is a subset of $\mathcal{M}_{2\times2}$ and the operations are the same. Now, for applying Theorem 0.7, take

$$\begin{bmatrix} a_1 & 0 \\ b_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ b_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ b_1 + b_2 & b_1 + b_2 \end{bmatrix} \in W$$

and

$$\alpha \begin{bmatrix} a_1 & 0 \\ b_1 & b_1 \end{bmatrix} = \begin{bmatrix} \alpha\, a_1 & 0 \\ \alpha\, b_1 & \alpha\, b_1 \end{bmatrix} \in W$$

as the resultant obtained in each case is of the form of elements in W. Hence W is a subspace of V.

We know that if vectors are added, the resultant is another vector (property (1)). Similarly if a vector is multiplied with a scalar, the resultant is again a vector (property (6)). If we combine these operations, we get a 'linear combination' of vectors.

Definition 0.9. *Let V be a vector space, $u_1, u_2, \ldots, u_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$. Then the vector*

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$$

is called a linear combination of the vectors $u_1, u_2, \ldots, u_n$.

Example 0.10. Consider the vector space $V = \mathcal{M}_{2\times 2}$ and let

$$u = \begin{bmatrix} 1 & 0 \\ 5 & -2 \end{bmatrix}, v = \begin{bmatrix} 3 & -5 \\ 7 & 1 \end{bmatrix}$$

Then $w = \begin{bmatrix} -7 & 15 \\ -11 & -7 \end{bmatrix}$ is a linear combination of u and v since $w = 2\,u + (-3)\,v$.

As we can see, the linear combination of vectors in V is again a vector in V. So clearly the set of all linear combinations of vectors in V is a subset of V. Moreover, it is easy to show:

Theorem 0.11. *Let V be a vector space V and $u_1, u_2, \ldots, u_n$ be vectors in V. Then the set of all linear combinations of these vectors $u_1, u_2, \ldots, u_n$ is always a subspace of V.*

There is a special name for this subspace.

Definition 0.12. *Let V be a vector space and $u_1, u_2, \ldots, u_n$ be vectors in V. Then the set of all linear combinations of these vectors $u_1, u_2, \ldots, u_n$ is called the linear span (or Span) of $u_1, u_2, \ldots, u_n$.*

$$\textit{i.e., Span}(u_1, u_2, \ldots, u_n) = \{\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n : \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}$$

Example 0.13. Consider the vector space $V = \mathcal{M}_{2\times 2}$ and let

$$u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } Span\{u,\, v\} = \left\{\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\} = \left\{\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} :\ \alpha_1,\, \alpha_2 \in \mathbb{R}\right\},$$

which is actually the set of all diagonal matrices of order 2×2.

In vector spaces, there are some vectors which can span the whole vector space.

Definition 0.14. *We say that a vector space V is spanned by the vectors $u_1, u_2, \ldots, u_n$ in V if each vector of V can be written as a linear combination of these vectors. In such a case, we call $u_1, u_2, \ldots, u_n$ as a spanning set for V. Sometimes we say that $u_1, u_2, \ldots, u_n$ spans V.*

Method to check if a given set is a spanning set or not: Suppose a vector space V is given and we need to check if a set $\{u_1, u_2, \ldots, u_n\}$ spans V or not.

i. Take a general vector v of V.
ii. Write v as a linear combination of $u_1, u_2, \ldots, u_n$
iii. Write the system of equations that results from step ii.
iv. Write the system in matrix form $AX = b$
v. Write the augmented matrix $[Ab]$
vi. Reduce $[Ab]$ to its Echelon form.
vii. If $\rho(A) = \rho(Ab)$ always, the system is solvable and hence given vectors will span V.
viii. If $\rho(A) \neq \rho(Ab)$ in some situations, the system is not always solvable and hence the given vectors do not span V.

Example 0.15. We show that the vectors

$$u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, u_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

span the vector space $\mathcal{M}_{2\times 2}$. For this, take a general vector

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$. Then it is easy to see that

$$v = au_1 + bu_2 + cu_3 + du_4.$$

Thus every vector is a linear combination of u_1, u_2, u_3, u_4. Hence they span the vector space $\mathcal{M}_{2\times 2}$.

Example 0.16. Consider the vectors $(1,1,1),(1,1,2),(1,2,2)$ in $\mathbb{R}^3$. To show that this set spans $\mathbb{R}^3$, take a general element (x,y,z) of $\mathbb{R}^3$. Writing

$$(x,y,z) = \alpha_1(1,1,1) + \alpha_2(1,1,2) + \alpha_3(1,2,2),$$

we get a system of equations:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= x \\ \alpha_1 + \alpha_2 + 2\alpha_3 &= y \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= z \end{aligned}$$

The corresponding matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The augmented matrix of this system is

$$[Ab] = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 2 & y \\ 1 & 2 & 2 & z \end{bmatrix}$$

An Echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & x \\ 0 & 1 & 2 & z-x \\ 0 & 0 & 1 & y-x \end{bmatrix}$$

From this it is clear that $\rho(Ab) = \rho(A) = 3$ whatever be (x,y,z). Hence the system is always consistent. Hence the given three vectors span $\mathbb{R}^3$.

It should also be noted that there may be plenty of spanning sets for a vector space.

Now we discuss about certain relations between vectors of a vector space.

Definition 0.17. *We say that the vectors $u_1, u_2, \ldots, u_n$ are linearly dependent if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that*

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$$

with <u>atleast</u> one scalar $\alpha_i \neq 0$.

This definition says that there must be a 'non-trivial' linear relationship between the vectors $u_1, u_2, \ldots, u_n$ if they are to be linearly dependent. It also implies that one of the vectors can be written as a linear combination of the other vectors (see problem 43). It happens if and only if one of the vectors is in the span of the other vectors (see problem 44).

Example 0.18. We can see that in $\mathcal{M}_{2\times 2}$, the vectors

$$u_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, u_2 = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$$

are linearly dependent since

$$\begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix} = 2 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},$$

or $1.u_1 + (-2).u_2 = 0$ holds here and $\alpha_1 = 1$ and $\alpha_2 = -2$ are both non zero here.

If a set of vectors fails to be linearly dependent, there will not be such a 'non-trivial' linear relation between them. In such a case, we call the vectors as linearly independent vectors.

Definition 0.19. *A set of vectors $u_1, u_2, \ldots, u_n$ in a vector space V is said to be linearly independent if whenever $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$ happens, then each of the scalars $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$. i.e.,*

vectors $u_1, u_2, \ldots, u_n$ are independent if and only if

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0 \quad \Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$$

Following is the definition of linear independence pertaining to an infinite set of vectors.

Definition 0.20. *An infinite set $\{u_1, u_2, \ldots\}$ of vectors is linearly independent if any collection of finitely many elements taken from the set is always linearly independent.*

Example 0.21. Consider the vectors

$$u_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$. Writing their linear combination equal to zero, i.e., $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$, we get a system of equations $2\,\alpha_1 + \alpha_2 = 0;\quad 2\alpha_2 + \alpha_3 = 0;\quad 2\alpha_3 = 0;\quad \alpha_1 = 0$. Solving these, we get each of the scalars equal to 0. Hence the three vectors are linearly independent.

There are easy methods to check linear independence in the case of some special vector spaces. The key point used in these methods is that the rank of a matrix is equal to the number of linearly independent rows of the matrix. Also we sometimes use the fact that, for an $n \times n$ square matrix, the rank is equal to n if and only if determinant is equal to zero.

Methods to check linear independence of vectors in the case of $\mathcal{P}, \mathcal{M}_{2\times 2}$ or $\mathbb{R}^n$:

- Form a matrix using the coefficients/coordinates of the given vectors
- Find the rank of this matrix using Echelon form (or any other method).
- If the rank is equal to the number of given vectors, the given vectors are independent; and if not, then they are dependent.

  (If the matrix thus obtained is a **square** matrix, one can find the determinant. Now if the determinant is non zero, the vectors should be independent).

One can do it staight from the definition. Write a linear combination of the vectors, put it equal to zero. Check if all scalars are becoming zero (independent) or not (dependent). (Eventhough this is a theoretically approved method for all vector spaces, the previous method will be easier in the case of $\mathcal{P}, \mathcal{M}_{2\times 2}$ or $\mathbb{R}^n$ under usual operations.)

Example 0.22. Consider the vectors $(1, 3, 4), (2, 4, 6), (1, 1, 5)$ in $\mathbb{R}^3$. To check if linearly independent, form the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$$

Now find the Echelon form, say

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

which shows that $\rho(A) = 3$. Hence the three vectors are linearly independent.

[Or else, since A is a square matrix, we find the determinant $|A| = -6 \neq 0$. Hence $\rho(A) = 3$ and hence the three vectors are linearly independent].

Example 0.23. We show that the vectors

$$u_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

are linearly independent in $\mathcal{M}_{2\times 2}$. For this, we form the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$. It Echelon form is of the form

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Hence $\rho(A) = 3$. Thus the three vectors are linearly independent.

Example 0.24. Consider the vectors $1 + 3t + 4t^2, 2 + 4t + 6t^2, 1 + t + 2t^2$ in $\mathcal{P}_2$. To check if linearly independent, form the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Now find the Echelon form, say

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which shows that $\rho(A) = 2 \neq 3$. Hence the three vectors are linearly dependent.

[Or else, since A is a square matrix, we find the determinant $|A| = 0$. Hence $\rho(A) \neq 3$ and hence the three vectors are linearly dependent].

There are some sets of vectors which can span the total vector space and at the same time, can be linearly independent. Such sets need special attention.

Definition 0.25. *A set of vectors in a vector space is called a basis for the vector space if the set is linearly independent and it spans the vector space.*

Example 0.26. Consider the vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ in $\mathbb{R}^3$. Clearly it is linearly independent, since its matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has rank 2. Now it spans $\mathbb{R}^3$ since any vector (a, b, c) can be written as the linear combination

$$(a, b, c) = \alpha_1(1, 0, 0) + \alpha_2(0, 1, 0) + \alpha_3(0, 0, 1)$$

with $\alpha_1 = a, \alpha_2 = b, \alpha_3 = c$. Thus it is a basis for $\mathbb{R}^3$.

Example 0.27. One can show that the set of vectors $\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

does not form a basis for $\mathcal{M}_{2\times 2}$.

As we have seen in Example 0.23, they are linearly independent in $\mathcal{M}_{2\times 2}$. But it can be shown that u_1, u_2, u_3 do not span all vectors of $\mathcal{M}_{2\times 2}$. Thus it is not a basis for $\mathcal{M}_{2\times 2}$.

Example 0.28. Consider the vectors $1 + 3t + 4t^2, 2 + 4t + 6t^2, 1 + t + 2t^2$ in $\mathcal{P}_2$. This collection is not a basis for $\mathcal{P}_2$ since they are not linearly independent as seen in Example 0.24.

Note: Some vector spaces have some special simple bases as in the following. Such bases as reffered as "Standard bases".

- The standard basis for $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
- The standard basis for $\mathbb{R}^n$ is $\{(1, 0, 0, \ldots, 0, 0), (0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1)\}$.
- The standard basis for $\mathcal{P}_n$ is $\{1, x, x^2, x^3, \ldots, x^n\}$
- The standard basis for $\mathcal{P}$ is $\{1, x, x^2, x^3, \ldots\}$
- The standard basis for $\mathcal{M}_{2\times 3}$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

There may be plenty of bases for the same vector space.

Example 0.29. We have seen in Example 0.26 that $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$. Now, the set $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ is also linearly independent and spans $\mathbb{R}^3$. Hence this set is also a basis for $\mathbb{R}^3$.

The following is an important result about the number of elements in a basis.

Theorem 0.30. *Any two bases for the same vector space will have equal number of elements.*

This result says that even though the elements in bases of the same vector space may be different, the number of elements in any basis should be equal.

Example 0.31. We have seen in Example 0.26 that $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$. So any other basis of $\mathbb{R}^3$ should also contain 3 elements in it. Due to this reason, any given subset of $\mathbb{R}^3$ which contains 2 or 4 elements can never be a basis for $\mathbb{R}^3$. But it should be noted that a 3 elemented subset in $\mathbb{R}^3$ may or may not be its basis of $\mathbb{R}^3$, as in this case we must check linear independence and spanning properties of the given set.

In a similar way, we can say that the set $\{1 + t^2, 2 + 4t - t^2, 3 - t + 3t^2\}$ can never be a basis for $\mathcal{P}_3$ since any basis of $\mathcal{P}_3$ must contain 4 elements.

There is a special name for the number of elements in a basis.

Definition 0.32. *The cardinality (number of elements) of a basis for a vector space is called the dimension of the vector space. It is denoted by* $dim(V)$.

In view of the note after Example 0.28, we have:

- $dim(\mathbb{R}^3) = 3$
- $dim(\mathbb{R}^n) = n$
- $dim(\mathcal{P}_n) = n + 1$
- $dim(\mathcal{M}_{m\times n}) = mn$
- $\mathcal{P}$ is an infinite dimensional vector space.

Eventhough a basis needs to be linearly independent and span the vector space, if we are sure about the dimension of the vector space, then the following theorem eases our work greatly.

Theorem 0.33. *Suppose* $dim(V) = n$. *Then a set of* n *vectors in* V *is a basis if and only if it is linearly independent.*

Note: So, when we know the dimension of the vector space is n, then to check whether a given subset is a basis or not, it is very easy.

step: i. Check if the given set has n vectors or not.

(If it does not have n vectors, then the given collection can never be a basis for the vector space as by theorem 0.30, every basis of the vector space whose dimension is n, must have n elements).

step: ii. If there are n vectors, test if the given n vectors are linearly independent or not.

(If it is not linearly independent, then it can never be a basis for the vector space because of Theorem 0.33).

step: iii. If both steps i and ii have positive answers, the set is a basis. Else, it is not.

Example 0.34. Consider the subset $\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} u_3 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

in $\mathcal{M}_{2\times 2}$.

We know that $dim(M_{2\times 2}) = 4$. Hence the given set can not be a basis as it contains only 3 elements.

Example 0.35. Consider the vectors $(1, 0, 0), (0, 1, 1), (1, 1, 1)$ in $\mathbb{R}^3$. It is not linearly independent since its corresponding matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

has rank 2. Hence it is not a basis for $\mathbb{R}^3$.

Example 0.36. Consider the vectors $1 + 3t + 4t^2, 2 + 4t + 6t^2, 1 + t + t^2$ in $\mathcal{P}_2$. Since $dim(\mathcal{P}_2) = 3$, it is enough to check the linear independence of these three vectors. On checking one can easily see that this set is linearly independent and hence by Theorem 0.30 it is a basis for $\mathcal{P}_2$.

We know that a spanning set, by definition, spans the total vector space. Hence each vector can be written as a linear combination of the spanning set. But the representation may not be in a unique way:

Example 0.37. Consider the vectors $(1, 0), (0, 1), (1, 1)$ in $\mathbb{R}^2$. It can be seen that this set spans $\mathbb{R}^2$. Also each element of $\mathbb{R}^2$ can be represented in many ways using these three vectors. For example,

$$(2, 3) = 2(1, 0) + 3(0, 1) + 0(1, 1)$$
$$(2, 3) = 0(1, 0) + 1(0, 1) + 2(1, 1)$$

But one can see that the set is not linearly independent. In otherwords it is a spanning set which is not linearly independent.

When the spanning set is also linearly independent, it becomes a basis. There is 'uniqueness' when we span a vector using a basis. In the following, by an ordered basis, we mean a basis in which the order of the vectors has importance (like an ordered pair).

Theorem 0.38. *Let $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a vector space V. Then each element of V can be written uniquely as a linear combination of elements of $\mathcal{B}$.*

Hence, given a basis $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ and a vector v in V, then we can write

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n,$$

and these scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ will be unique.

Definition 0.39. *Let V be a vector space and $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for V. Then the coordinates of a vector v with respect to $\mathcal{B}$ (denoted by $[v]_{\mathcal{B}}$) is the column vector $[\alpha_1, \alpha_2, \dots, \alpha_n]^T$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the unique scalars such that*

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

Example 0.40. Consider the set $\mathcal{B} = \{(1,0,0), (1,1,0), (1,1,1)\}$ in $\mathbb{R}^3$. It is a basis for $\mathbb{R}^3$. Now we shall find the coordinate matrix of $(2,3,-1)$ with respect to $\mathcal{B}$. Writting

$$(2,3,-1) = \alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1),$$

we get $\alpha_1 = -1, \alpha_2 = 4, a_3 = -1$. Hence the coordinate matrix $[(2,3,-1)]_{\mathcal{B}} = [-1 \quad 4 \quad -1]^T$.

Note: When we take different bases $\mathcal{B}_1$ and $\mathcal{B}_2$, the coordinates of the same vector can be different with respect to them! (i.e., $[v]_{\mathcal{B}_1} \neq [v]_{\mathcal{B}_2}$).

Now, if we take two different bases for a vector space, we can find the coordinate vectors of each element of one basis with respect to the other basis. Using these column vectors, one can form a new matrix.

Definition 0.41. *Suppose V be a vector space and $\mathcal{B}_1 = \{u_1, u_2, \dots, u_n\}$ and $\mathcal{B}_2 = \{v_1, v_2, \dots, v_n\}$ be two ordered bases for V. Form an $n \times n$ matrix by taking $[u_1]_{\mathcal{B}_2}, [u_2]_{\mathcal{B}_2}, \dots, [u_n]_{\mathcal{B}_2}$ as columns. The matrix thus obtained is called the* **transition matrix** *corresponding to the change of basis $\mathcal{B}_1 \to \mathcal{B}_2$ and is denoted by $[T]_{\mathcal{B}_1 \to \mathcal{B}_2}$.*

When we consider V with $\mathcal{B}_1$ as its basis, every element has a representation and coordinate matrix with respect to this basis. Suppose suddently one wants to study V with a second basis $\mathcal{B}_2$. Then he has to change all the representations and coordinate vectors as everything changes when the basis changes. But with the help of the transition matrix, one can easily get the coordinate matrix of elements with respect to $\mathcal{B}_2$ using the coordinate matrix of elements with respect to $\mathcal{B}_1$. One can show that

$$[v]_{\mathcal{B}_2} = [T]_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_1}$$

Also one can go back to the settings of V with $\mathcal{B}_1$ by using the inverse of the transition matrix:

$$[v]_{\mathcal{B}_1} = [T]^{-1}_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_2}.$$

This is why transition matrix *corresponds to "change of basis"*.

Example 0.42. The sets $\mathcal{B}_1 = \{(1,0,0), (1,1,0), (1,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$ are bases for $\mathbb{R}^3$. representing each element of $\mathcal{B}_2$ with respect to $\mathcal{B}_1$, we get

$$\begin{aligned} [(1,0,0)]_{\mathcal{B}_2} &= [1 \quad 0 \quad 0]^T \\ [(1,1,0)]_{\mathcal{B}_2} &= [1 \quad 1 \quad 0]^T \\ [(1,1,1)]_{\mathcal{B}_2} &= [1 \quad 1 \quad 1]^T \end{aligned}$$

Hence the transition matrix corresponding to the change of basis $\mathcal{B}_1 \to \mathcal{B}_2$ is given by

$$[T]_{\mathcal{B}_1 \to \mathcal{B}_2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 0.43. The sets $\mathcal{B}_1 = \{(1,0,0), (1,1,0), (1,1,1)\}$ and $\mathcal{B}_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$ are bases for $\mathbb{R}^3$. representing each element of $\mathcal{B}_2$ with respect to $\mathcal{B}_1$, we get

$$\begin{aligned} [(1,0,0)]_{\mathcal{B}_1} &= [1 \quad 0 \quad 0]^T \\ [(0,1,0)]_{\mathcal{B}_1} &= [-1 \quad 1 \quad 0]^T \\ [(0,0,1)]_{\mathcal{B}_1} &= [0 \quad -1 \quad 1]^T \end{aligned}$$

Hence the transition matrix corresponding to the change of basis $\mathcal{B}_2 \to \mathcal{B}_1$ is given by

$$[T]_{\mathcal{B}_2 \to \mathcal{B}_1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

One may observe that for any $v \in V$,

$$[v]_{\mathcal{B}_2} = [T]_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_2}$$

and

$$[v]_{\mathcal{B}_1} = [T]_{\mathcal{B}_2 \to \mathcal{B}_1} [v]_{\mathcal{B}_1}.$$

Also one can see that $[T]_{\mathcal{B}_2 \to \mathcal{B}_1}$ is the inverse of the matrix $[T]_{\mathcal{B}_1 \to \mathcal{B}_2}$.

Inner Product Spaces

Suppose that V is a vector space over $\mathbb{R}$. We have seen that there are two operations defined in connection with a vector space - the vector addition and scalar multiplication.

Now we talk about a new operation namely, 'taking the inner product of two vectors' on a vector space V. It helps to introduce the concept of *angle* and *orthogonality* (or *perpendicularity*) in the vector space settings. It is also an extension of the concept of *'dot product'* of 3-dimensional real vectors (considering $\mathbf{a}i + \mathbf{b}j + \mathbf{c}k$ as (a, b, c)) to a general vector space.

Definition 0.44. *A vector space V is said to be an* **inner product space** *if there is defined a function $\langle ., . \rangle : V \times V \to \mathbb{R}$ satisfying the following properties:*

1. $\langle u, u \rangle \geq 0$ *for all* $u \in V$
2. $\langle u, u \rangle = 0$ *if and only if* $u = 0$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ *for all* $u, v, w \in V$.
4. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ *for all* $u, v \in V$ *and* $\alpha \in \mathbb{K}$.
5. $\langle v, u \rangle = \langle u, v \rangle$ *for all* $u, v \in V$.

The above defined function $\langle ., . \rangle$ on V is called an **inner product** *on V.*

The properties 1 and 2 are combinedly called **positive definiteness** and the properties 3 and 4 together are called **linearity properties**. Property 5 is called **symmetry**. It should be noted that on all vector spaces, we may not be able to define inner products.

We use the notations $X, Y, Z, \ldots$ for inner product spaces (instead of $U, V, W, \ldots$).

Example 0.45.

1. Take $X = \mathbb{R}$ and define
$$\langle x, y \rangle = xy,$$
the usual multiplication of real numbers.
2. Take $X = \mathbb{R}^3$ and define
$$\langle (a_1, b_1, c_1), (a_2, b_2, c_2) \rangle = a_1 a_2 + b_1 b_2 + c_1 c_2$$
(the usual 'dot product' of real 3-dimensional vectors).
3. Take $X = \mathbb{R}^n$ and define
$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = \sum_{j=1}^{n} x_j y_j$$
5. Take $X = P_n[a, b]$ and define
$$\langle p(x), q(x) \rangle = \int_a^b p(x) q(x) dx$$

Here $\mathcal{P}_n[a, b]$ denotes the set of all real polynomials of degree less than or equal to n whose domain are considered as $[a, b]$.

Note that the inner product on an inner product space gives rise to a positive quantity
$$\|x\| = \sqrt{\langle x, x \rangle}$$
which is called the **norm** of the vector x in X. When the norm of a vector is 1, we say that the vector is a **normalized vector**.

We know that in the case of dot product of real 3-dimensional vectors,
$$a.b = |a||b| cos\theta,$$
where θ is the angle between the vectors a and b. Also, we say that two vectors a and b are *perpendicular* if the angle between them is $\frac{\pi}{2}$, i.e, if $a.b = 0$. Similar to this we define orthogonality of vectors in inner product space.

Definition 0.46. *We say that $x, y \in X$ are* **orthogonal** *to each other (written as $x \perp y$) if*
$$\langle x, y \rangle = 0.$$
We say that a set of vectors $\{x_1, x_2, \ldots, x_n\}$ in X is an **orthogonal set** *if each pair of distict vectors of the set is orthogonal,i.e.,*
$$\langle x_i, x_j \rangle = 0 \text{ whenever } i \neq j.$$

Example 0.47. *The vectors* $(1,0,1)$ *and* $(1,0,-1)$ *are orthogonal in* $\mathbb{R}^3$*.*

Definition 0.48. *We say that a set of vectors* $\{x_1, x_2, \ldots, x_n\}$ *in* X *is an* **orthonormal set** *if it is an orthogonal set and the norm of each element of the set is* 1*. That is*

$$\langle x_i, x_j \rangle = 0 \text{ whenever } i \neq j \text{ and } \langle x_i, x_i \rangle = 1 \text{ for all } i$$

.

Example 0.49. *The vectors* $(1,0,1)$ *and* $(1,0,-1)$ *are orthogonal in* $\mathbb{R}^3$ *but not orthonormal; same is the case with* $1+x^2$ *and* $x+x^3$ *in* $\mathcal{P}_3[-1,1]$*. But the set* $\{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}})\}$ *is an orthonormal set in* $\mathbb{R}^3$*.*

Note that if $\{x_1, x_2, \ldots\}$ is orthogonal in X, then $\{\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \ldots\}$ will be an orthonormal set in X (use $\|\alpha x\| = |\alpha| \|x\|$ and take $\frac{1}{\|x\|}$ as the scalar, to show this). Thus in this way, one can *orthonormalize* any given orthogonal set.

Recall that two vectors in 3-dimensional space is said to be perpendicular if the dot product of the vectors is zero. This happens if and only if the angle between them is $\pi/2$. Corresponding to the concept of angle between two vectors in 3-dimensional space, one may define the angle between two vectors in an inner product space as

$$cos\, \theta = \frac{\langle x, y \rangle}{\sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}}.$$

Eventhough this is just a generalization, since $|cos\, \theta| \leq 1$, one may wonder if

$$\frac{|\langle x, y \rangle|}{\sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}} \leq 1$$

happens in a general inner product space! A positive answer was given by Cauchy and Schwartz:

Theorem 0.50. *Suppose* X *is an inner product space and* $x, y \in X$*. Then these vectors satisfy the so called* **Cauchy-Schwartz Inequality***:*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

Problem: Verify Schwartz inequality for $x = (2,1,0)$ and $y = (1,3,-1)$ in $\mathbb{R}^3$.

When two vectors of a triangle are perpendicular in $\mathbb{R}^2$, one knows that

$$\text{base}^2 + \text{altitude}^2 = \text{hypotenus}^2.$$

A similar result holds for any two orthogonal vectors in any inner product space.

Theorem 0.51. *Suppose* X *is an inner product space and* x, y *are* **orthogonal** *in X. Then these vectors satisfy the so called* **Pythagorus Theorem***:*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: Start with $\|x+y\|^2 = \langle x+y,\, x+y \rangle$ and use $\langle x, y \rangle = \langle y, x \rangle = 0$ on expansion.

Problem: Verify Pythagorus theorem for $x = (2,1,0)$ and $y = (3,-6,5)$ in $\mathbb{R}^3$.

Definition 0.52. *We say that a set of vectors* $\{x_1, x_2, \ldots, x_n\}$ *in* X *is an* **orthogonal basis** *for* X *if it is a basis for* X *and its elements are orthogonal. Similarly we define* **orthonormal basis** *(orthonormal set, which is also a basis).*

Problem: Check if the set $(1,0,0), (0,1,0), (0,0,1)$ is an orthonormal basis for $\mathbb{R}^3$.

Theorem 0.53. *Every orthogonal set in an inner product space is linearly independent.*

Proof: Let $\{u_1, u_2, \ldots, u_n\}$ be a linearly independent set in an inner product space. Put $c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0$, where c_i are scalars. Taking inner product with u_i on both sides and and using $\langle u_i, u_j \rangle = 0$ and $\langle u_i, u_j \rangle \neq 0$, one can show that $c_i = 0$. Do this for $i = 1, 2, \ldots, n$.

Problem: Show that $\{(1,2), (1,3)\}$ is a linearly independent set in the inner product space $\mathbb{R}^2$, whereas they are not orthogonal.

The above problem shows that a linearly independent set in an inner product space need not be orthogonal (eventhough an orthogonal set must be linearly independent). The following result shows that from every linearly independent set, we can construct an orthonormal set through a process called *Gramm-Schmidt orthonormalization theorem.*

Theorem 0.54. *Let $\{y_1, y_2, \ldots\}$ be a linearly independent set in X. Then by* **Gramm-Schmidt orthonormalization process***:*

$$\begin{aligned}
u_1 &= y_1; \\
u_2 &= y_2 - \frac{\langle y_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1; \\
u_3 &= y_3 - \frac{\langle y_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1 - \frac{\langle y_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2; \\
\ldots & \ldots \ldots \\
u_n &= y_n - \sum_{i=1}^{n-1} \frac{\langle y_n, u_i\rangle}{\langle u_i, u_i\rangle} u_i; \\
\ldots & \ldots \ldots
\end{aligned}$$

one can create the set $\{u_1, u_2, u_3, \ldots\}$, which will be an orthogonal set and the set

$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}, \ldots \right\}$$

will be an orthonormal set in X.

One can always create orthonormal set in a nonzero inner product space by taking a linearly independent set and applying Gramm-Schmidt orthonormalization.

Example 0.55. The vectors $y_1 = (1, 1, 0), y_2 = (1, 0, 0), y_3 = (1, 1, 1)$ are independent in $\mathbb{R}^3$ (basis). On applying Gramm-Schmidt orthogonalization, we get

$$u_1 = (1, 1, 0), \;\; u_2 = (\frac{1}{2}, -\frac{1}{2}, 0), \;\; u_3 = (0, 0, 1),$$

which is orthogonal. Normalizing it we get

$$u_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \;\; u_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right), \;\; u_3 = (0, 0, 1).$$

which is the orthonormal set (basis).

1. **Problems**

(1) *Show that the set of all $m \times n$ matrices over $\mathbb{R}$ with usual matrix addition and scalar multiplication of a matrix with a real number is a vector space.*
(2) *Show that the set of all $n \times n$ diagonal matrices with usual matrix addition and scalar multiplication of a matrix with a real number is a vector space.*
(3) *Show that the set of all 3×3 matrices whose second and third rows are zeros is a vector space under the usual matrix addition and scalar multiplication.*
(4) *Show that the set of all $n \times n$ non-singular matrices is not a vector space under usual matrix addition and scalar multiplication of a matrix with a real number.*
(5) *Show that the set of all real polynomials of degree less than n is a vector space under the usual addition of polynomials and the multiplication of a polynomial with a real number.*
(6) *Show that the set of all real polynomials is a vector space under the usual addition of polynomials and the multiplication of a polynomial with a real number.*
(7) *Show that the set of all real valued functions on $[a, b]$ is a vector space under the operations:*

$$(f + g)(x) = f(x) + g(x), \quad x \in [a, b]$$
$$(\alpha.f)(x) = \alpha f(x), \quad x \in [a, b].$$

(8) *Show that the set $\mathbb{R}^n$ of all $n-$tuples forms a vector space under the operations*

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\text{and } \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha, x_2, \ldots, \alpha x_n).$$

(9) *Show that the set of all ordered triplets of the form $(x, 0, z), \quad x, y \in \mathbb{R}$ is a vector space under the usual addition $(x_1, 0, z_1)+(x_2, 0, z_2) = (x_1+x_2, 0, z_1+z_2)$ and scalar multiplication $\alpha(x_1, 0, z_1) = (\alpha x_1, 0, \alpha z_1)$.*
(10) *On $V = \mathbb{R}^3$, define*

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + 2b_2, c_1 + 3c_2)$$

and

$$\alpha(a_1, b_1, c_1) = (\alpha a_1, 2\alpha b_1, 3\alpha c_1).$$

Is V a vector space under these operations?
(11) *Is $V = \mathbb{R}^2$ with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1)$ and $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$ a vector space?*
(12) *Is $V = \mathbb{R}^3$ with the operations $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ and $\alpha(x_1, y_1, z_1) = (\alpha x_1, y_1, z_1)$ a vector space?*
(13) *Test if $V = \mathbb{R}$ is a vector space with vector addition defined as $u - v$ and the usual scalar multiplication.*
(14) *Show that in any vector space V,*
  (i) $-(-u) = u$
  (ii) $cu = 0$ *if and only if* $c = 0$ *or* $u = O$ *or both.*
  (iii) $c.0 = 0$.
(15) *Show that a vector space has only one identity vector.*
(16) *Is $V = \{at^2 + bt + c : b = 2a + 3\}$ a subspace of $P_2(t)$?*
(17) *Is*

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \quad \& \quad b = c \right\}$$

a subspace of $\mathcal{M}_{2\times2}$?
(18) *Is*

$$W = \left\{ \begin{bmatrix} a & b \\ a + b & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

a subspace of $\mathcal{M}_{2\times2}$?
(19) *Show that the set of all vectors in $\mathbb{R}^3$ satisfying $x_1 - 2x_2 + 4x_3 = 0$ is a subspace of $\mathbb{R}^3$.*
(20) *Show that the set of all $n \times n$ symmetric matrices is a subspace of $\mathcal{M}_{n\times n}$.*
(21) *Show that the set of all 2×3 matrices with all entries non negative, is not a subspace of $\mathcal{M}_{2\times3}$.*
(22) *Check if the set of all 3×3 skew-symmetric matrices is a subspace of $\mathcal{M}_{3\times3}$.*
(23) *Check if the set of all 3×3 singular matrices is a subspace of $\mathcal{M}_{3\times3}$.*
(24) *Check if the set of all 3×3 non-singular matrices is a subspace of $\mathcal{M}_{3\times3}$.*
(25) *Check if the set of all polynomials whose degree less than or equal to 3 and having positive coefficients is a subspace of $\mathcal{P}_3$.*
(26) *Show that the set of all functions of the form $a \cos x + b \sin x$, with any constants $a, b \in \mathbb{R}$ is a subspace of the vector space given in problem (7).*
(27) *Check if the set of all $3-$tuples of the form $(a, a - b, a + 3b) : a, b \in \mathbb{R}$, is a subspace of $\mathbb{R}^3$.*

(28) *Check if the set of all* $3-$*tuples of the form* $(a,b,c): a=b,$ *is a subspace of* $\mathbb{R}^3$.
(29) *Check if the set of all* $4-$*tuples of the form* $(a,b,c,d): a^2=b^2,$ *is a subspace of* $\mathbb{R}^4$.
(30) *Check if the set of all* $4-$*tuples of the form* $(a,b,c,d): a+b=c+d,$ *is a subspace of* $\mathbb{R}^4$.
(31) *Check if the set of all* $3-$*tuples of the form* $(a,b,a,b),$ *is a subspace of* $\mathbb{R}^3$.
(32) *Check if the set of all* $3-$*tuples of the form* $(a,b,c): 4a-3b+c=5,$ *is a subspace of* R^3.
(33) *Check if the set of all* $3-$*tuples of the form* $(a,b,c,d): a^3=b^3,$ *is a subspace of* $\mathbb{R}^3$.
(34) *Check if the set of all matrices of the form a a subspace of* $\mathcal{M}_{2\times 2}$.
(35) *Check if the set* $\{(a,b,1): a,b\in\mathbb{R}\}$ *is a subspace of* $\mathbb{R}^3$.
(36) *Check if the set* $\{(a,b,0): a,b\in\mathbb{R}\}$ *is a subspace of* $\mathbb{R}^3$.
(37) *Check if the set* $\{(a,b,c): a\geq 0, b\geq 0, c\geq 0\}$ *is a subspace of* $\mathbb{R}^3$.
(38) *Check if the set* $\{(x,y,z): z=y+x\}$ *is a subspace of* $\mathbb{R}^3$.
(39) *Check if the set* $\{(a,b,c): b=a-c\}$ *is a subspace of* $\mathbb{R}^3$.
(40) *Check if the set* $\{(a,b,c): a=0, c=0\}$ *is a subspace of* $\mathbb{R}^3$.
(41) *Check if the set* $\{(a,b,c): b=3\}$ *is a subspace of* $\mathbb{R}^3$.
(42) *Check if the set* $\{(x,y)\in\mathbb{R}^2: 3x+4y=0\}$ *is a subspace of* $\mathbb{R}^3$.
(43) *Show that if a set* $\{u_1,u_2,\ldots,u_n\}$ *is linearly dependent if and only if one of the vectors is a linear combination of the remaining vectors.*
(44) *Show that if a set* $\{u_1,u_2,\ldots,u_n\}$ *is linearly dependent if and only if one of the vectors is in the span of the remaining vectors.*